Implicitization of curves and surfaces using predicted support

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Outline

1. Introduction: some definitions

2. Newton polytope of the resultant

3. Implicitization by linear algebra

4. Results of the experiments

5. Conclusions and the future work
1. Introduction: some definitions

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Problem of the Implicitization

A **parameterization** of a geometric object in a space of dimension $n$ can be described by a set of parametric equations:

$$x_0 = f_0(t_1, ..., t_n), ..., x_n = f_n(t_1, ..., t_n),$$

where $t_1, t_2, ..., t_n$ are the parameters.

Performing **implicitization** means finding polynomial implicit equation

$$p(x_0, ..., x_n) = 0,$$

s.t.

$$p(f_0(t_1, ..., t_n), ..., f_n(t_1, ..., t_n)) = 0$$

for all the $t_1, t_2, ..., t_n$. 
Problem of the Implicitization

Implicitization is reduced to computing the **resultant** of

\[ x_i - f_i(t) \]

seen as polynomials in \( t \).

Same is valid for the curves and surfaces of rational parametrisation:

\[ x_i = \frac{g_i(t)}{h_i(t)} \]

Represented as polynomials in \( t \):

\[ x_i h_i(t) - g_i(t) = 0 \]
Newton Polytope

▶ The **support** of a polynomial

\[ f_i = \sum_j c_{ij} t^{a_{ij}} \]

is the set:

\[ \text{sup}(f_i) = \{ a_{ij} \in \mathbb{N}^n : c_{ij} \neq 0 \} \]

▶ Given a polynomial \( f_i \) its **Newton polytope** \( N(f_i) \) is the convex hull of its support.

\[ f_0 = x^2 y^3 + 3x - 5y^2 \]

\[ f_1 = x^3 - x^2 + y^2 x - 3y \]
Resultant polytope

Let $f = \{f_0, f_1, \ldots, f_n\}$ a polynomial system where $f_i \in K[t_1, \ldots, t_n]$ with coefficients $c_{ij}$.

- The **resultant** of $f$ is a polynomial $R \in K[c_{ij}]$ s.t. $R = 0$ iff $f$ has a common root.

- The Newton polytope of the Resultant polynomial $N(R)$ is called the **Resultant polytope**.

- We call an **extreme term** of $R$ a monomial which correspond to a vertex of $N(R)$.

**Example**

\[ f_0 = c_{00} + c_{01}t \]
\[ f_1 = c_{10} + c_{11}t \]
\[ R = c_{00}c_{11} - c_{01}c_{10} \]

\[(0, 1, 1, 0)\]
\[(1, 0, 0, 1)\]
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4. Results of the experiments

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Implicit Newton polytope

Consider parameterizations with fixed supports.

- **Generic** coefficients:
  - Compute the resultant’s Newton polytope, then specialize:
    [E-Kotsireas’03] developed Maple code calling Topcom [Rambau];
  - Tropical geometry for varieties of codim > 1.
    For curves, specified implicit polygon [Sturmfels-Tevelev-Yu’07].

- **Arbitrary** coefficients:
  - Implicit Newton polygon for curves:
    [Dickenstein-Feichtner-Sturmfels’07] study tropical discriminants;
    [D’Andrea-Sombra’07] use mixed fiber polytopes
    [Esterov-Khovanskii’07].
**Algorithm to compute the Resultant polytope**

Implemented by Vissarion Fisikopoulos [Emiris-Konaxis-Fisikopoulos’10].

**Input:** A polynomial system \( f = \{f_0, f_1, \ldots, f_k\} \)

**Output:** The Resultant polytope of \( f \): \( N(R) \)

1. Compute the Newton polytopes of the supports of \( f \): \( A_0, \ldots, A_k \)

2. Compute the Cayley embedding \( C(A_0, \ldots, A_k) \)

3. Enumerate all regular triangulations of \( C(A_0, \ldots, A_k) \).
   *i.e. the regular fine mixed subdivisions of \( A_0 + \cdots + A_k \)*

4. For each regular fine mixed subdivision compute the corresponding extreme term of \( R \)
Theorem [Sturmfels]

Given a polynomial system $f$ and a regular fine mixed subdivision of the Minkowski sum of the Newton polytopes of its supports we get an extreme term of the resultant $R$ which is equal

$$const \cdot \prod_{i=0}^{k} \prod_{\sigma} c_{iF_i}^{vol(\sigma)}$$

where $\sigma = F_0 + F_1 + \cdots + F_k$ is an $i$-mixed cell and $const \in \{-1, +1\}$. 
1. Introduction: some definitions

2. Newton polytope of the resultant

3. Implicitization by linear algebra

4. Results of the experiments

5. Conclusions and the future work
Implicitization algorithm

Parametric polynomial expressions

\[ x_i - f_i(t) \in K[t], i \in [0, n] \]

Support

Compute extremal terms of the Resultant Polytope

Compute inner points of the Resultant Polytope

Implicit equation support

Find coefficients by solving linear system

Coefficients (in relation to \( t \))

Implicit equation
Implicitization of the curve

**Witch of Agnesi**: parametric expressions

\[ x = at, \quad y = \frac{a}{1 + t^2} \]

Represented as polynomials: \(-x + at = 0, \ a - y - yt^2 = 0\)

Coefficients: \(c_{00} = -x; \ c_{01} = a; \ c_{10} = a - y; \ c_{11} = -y\)

Support of the parametric equations: \([0, 1, 0, 2]\)
Implicitization of the curve

Support of the implicit equation: \([0, 2, 1, 0], [2, 0, 0, 1]\)
In this case Resultant Polynomial consists only of two 4-dimensional points and has no inside points.

Form matrix \(M\): \(x_0, x_1, y_0, y_1\) are values of \(x, y\) obtained by using (random) values for the parameter \(t\).

\[
M = \begin{bmatrix}
c_{01}^2c_{10} & c_{00}^2c_{11} \\
c_{01}^2c_{10} & c_{00}^2c_{11}
\end{bmatrix} \Rightarrow \begin{bmatrix} a^3 - a^2y_0 & -x_0^2y_0 \\ a^3 - a^2y_1 & -x_1^2y_1 \end{bmatrix}
\]

and solve linear system \(M \cdot r = 0\) to find the coefficients \(r\).

Implicit equation:

\[
r_0c_{01}^2c_{10} + r_1c_{00}^2c_{11} = 0
\]

Result: \(a^2y + x^2y = a^3\).
Implicitization of the surface

**Cylinder**: parametric representation \( x = \cos(u), y = \sin(u), z = v \)

Polynomials: \( 1 - xt^2 - x - t^2 = 0, -y - yt^2 + 2t = 0, -z + s = 0. \)

Coefficients:

\[
\begin{align*}
    c_{00} &= 1 - x \\
    c_{01} &= -1 - x \\
    c_{10} &= -y \\
    c_{11} &= 2 \\
    c_{12} &= -y \\
    c_{20} &= -z \\
    c_{21} &= 1
\end{align*}
\]

Support of the parametric equations:

\[
\begin{pmatrix}
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
1 & 0 \\
2 & 0 \\
0 & 0 \\
0 & 1
\end{pmatrix}
\]
Implicitization of the surface

Support of the implicit equation:
\[ [0, 2, 2, 0, 0, 0, 0], [1, 1, 0, 2, 0, 0, 0], [1, 1, 1, 0, 1, 0, 0], [2, 0, 0, 0, 2, 0, 0] \]

\[
M = \begin{bmatrix}
    c_{01}^2 c_{10}^2 & c_{00} c_{01} c_{21}^2 & c_{00} c_{01} c_{10} c_{12} & c_{00}^2 c_{12} \\
    c_{01}^2 c_{10}^2 & c_{00} c_{01} c_{21}^2 & c_{00} c_{01} c_{10} c_{12} & c_{00}^2 c_{12} \\
    c_{01}^2 c_{10}^2 & c_{00} c_{01} c_{21}^2 & c_{00} c_{01} c_{10} c_{12} & c_{00}^2 c_{12} \\
    c_{01}^2 c_{10}^2 & c_{00} c_{01} c_{21}^2 & c_{00} c_{01} c_{10} c_{12} & c_{00}^2 c_{12}
\end{bmatrix}
\]

Solve linear system:
\[
[M] \begin{bmatrix} r_0 \\ r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
\]

Result: \( x^2 + y^2 - 1 = 0 \).
Implicitization by linear algebra

\( S = \) monomials forming \( \) a superset of \( \) the implicit support. \( C = \) unknown coefficients of implicit equation wrt \( S, \) \( |C| = |S|. \)

- \( MC = \vec{0}, \) where matrix \( M \) is \( |S| \times |S|, \) and contains values of \( S \) at points \((s_i, t_i), i = 1, \ldots, |S|\). Try roots of unity [Sturmfels-Tevelev-Yu’07].

- \( (SS^T)C = \vec{0}, \) substitute \( x, y, z \) by parametric expressions in \( K[s, t], \) integrate over \( s, t; \) solve for \( C \) [Corless-Galligo-Kotsireas-Watt’01].

Example: \( \text{supp}(H) \subset \{x^3y, x^3, x^3y^2, y^2z^3\}, \) then

\[
SS^T = \begin{bmatrix}
x^6y^2 & x^6y & x^6y^3 & x^3y^3z^3 \\
x^6y & x^6 & x^6y^2 & x^3y^2z^3 \\
x^6y^3 & x^6y^2 & x^6y^4 & x^3y^4z^3 \\
x^3y^3z^3 & x^3y^2z^3 & x^3y^3z^3 & y^4z^6
\end{bmatrix}
\Rightarrow C = \begin{bmatrix}
-2 \\
1 \\
1 \\
-1
\end{bmatrix}.
\]

- Approximate implicitization [Dokken].
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Examples: Curves

<table>
<thead>
<tr>
<th>Curves</th>
<th>Param. eq. degree.</th>
<th>Param. eq. terms</th>
<th>TOPCOM runtime, s</th>
<th>Resultant extremal terms</th>
<th>All resultant terms</th>
<th>Solv. time (Maple), s</th>
<th>Impl eq. degree</th>
</tr>
</thead>
<tbody>
<tr>
<td>Witch of Agnesi</td>
<td>1, 2</td>
<td>4</td>
<td>0.004</td>
<td>2</td>
<td>2</td>
<td>0.22</td>
<td>3</td>
</tr>
<tr>
<td>Circle</td>
<td>2, 2</td>
<td>5</td>
<td>0.004</td>
<td>3</td>
<td>4</td>
<td>0.48</td>
<td>2</td>
</tr>
<tr>
<td>Conhoid</td>
<td>2, 3</td>
<td>6</td>
<td>0.008</td>
<td>4</td>
<td>6</td>
<td>0.78</td>
<td>4</td>
</tr>
<tr>
<td>Folium of Descartes</td>
<td>3, 3</td>
<td>6</td>
<td>0.008</td>
<td>6</td>
<td>10</td>
<td>1.01</td>
<td>3</td>
</tr>
<tr>
<td>Cardioid</td>
<td>4, 4</td>
<td>7</td>
<td>0.024</td>
<td>10</td>
<td>33</td>
<td>1.88</td>
<td>4</td>
</tr>
</tbody>
</table>
Examples: Surfaces

<table>
<thead>
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<th>Param. eq. degree.</th>
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<th>TOPCOM runtime, s</th>
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<th>Solv. time (Maple), s</th>
<th>Impl eq. degree</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cylinder</td>
<td>2, 2, 1</td>
<td>7</td>
<td>0.006</td>
<td>3</td>
<td>4</td>
<td>0.88</td>
<td>2</td>
</tr>
<tr>
<td>Cone</td>
<td>3, 3, 1</td>
<td>9</td>
<td>0.288</td>
<td>8</td>
<td>14</td>
<td>1.19</td>
<td>2*</td>
</tr>
<tr>
<td>Paraboloid</td>
<td>3, 2, 2</td>
<td>9</td>
<td>0.296</td>
<td>8</td>
<td>37</td>
<td>1.44</td>
<td>2*</td>
</tr>
</tbody>
</table>

* In some cases, this approach gives, as implicit equation, the image under the Veronese mapping of the classical equation.
Implicitization experiments on curves and surfaces

These are some experimental results of the algorithms that compute the Newton polytope of the resultant. For more information see [1].

For the case of curves, there are two equations on one variable t. For the case of surfaces, there are three equations on two variables t,s. Additionally, a,b,c are constants. Each parametric equation can be transformed to a polynomial. The supports of the polynomials after the Cayley trick are stored in the files in the “supports” column. The file format is as follows: the first line contains the number of points and their dimension and in the following lines each column contains the coordinates of a point.

Curves

<table>
<thead>
<tr>
<th>No</th>
<th>Curve</th>
<th>Equation</th>
<th>Supports</th>
<th># Mixed Subdivisions</th>
<th>EXUM</th>
<th>TOPCOM PointZallTriang (SEC)</th>
<th>TOPCOM PointZTriang (SEC)</th>
<th># Mixed Cell Configurations</th>
<th># N(R) Vertices</th>
<th>N(R) Vertices</th>
<th># All Lattice Points N(R)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>astroid</td>
<td>( t \cos(t)^3 ) (</td>
<td>\sin(t)</td>
<td>^3 )</td>
<td>supports ( \square )</td>
<td>289</td>
<td>193.92</td>
<td>0.046</td>
<td>0.452</td>
<td>239</td>
<td>35</td>
</tr>
<tr>
<td>2</td>
<td>cardoid</td>
<td>( \frac{a}{2} - \cos(t) ) ( \cos(3t) ) ( a(2</td>
<td>\sin(t)</td>
<td>- \sin(2t)))</td>
<td>supports ( \square )</td>
<td>37</td>
<td>6.62</td>
<td>0.096</td>
<td>0.024</td>
<td>37</td>
<td>10</td>
</tr>
<tr>
<td>3</td>
<td>circle</td>
<td>( f \cos(t), \sin(t) )</td>
<td>supports ( \square )</td>
<td>5</td>
<td>0.004</td>
<td>0.016</td>
<td>0.004</td>
<td>5</td>
<td>3</td>
<td>N(R) ( \square )</td>
<td>4</td>
</tr>
</tbody>
</table>
Outline

1. Introduction: some definitions

2. Newton polytope of the resultant

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4. Results of the experiments

5. Conclusions and the future work
Conclusions and the future work

Conclusions:

▶ The problem of implicitization of geometric objects, parametrized by polynomials, can be reduced to solving a linear equations system by applying support prediction.
▶ We explore Newton polytope sparseness computing the polynomial support.

Future work prospects:

▶ Improve the support calculating algorithm [Emiris-Konaxis-Fisikopoulos’10].
▶ Improve linear system solving by choosing appropriate $t$ values to evaluate $x_i$ while forming matrix.
▶ Combine support prediction with approximate implicitization.
Thank You!